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PRINCIPLE HEISENBERG SUBALGEBRA IN A QUANTUM GROUP

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Abstract

We construct generators of the principle Heisenberg subalgebra in the quantized universal enveloping algebra $U(\widehat{sl_2})$. Applications for exactly solvable models are proposed.

1. Introduction

Quantum groups [1] are among central objects of interest both in pure mathematics and mathematical physics. Numerous applications of quantum group structure are known to have indispensable success in disclosure of underlying mathematical structures in problems of real-life physics and abstract models. Heisenberg subalgebras inside Kac-Moody Lie algebras [6] play a fundamental role in separating Heisenberg-type algebraic structures among huge sets of basises admitted for descriptions of physical models. Their importance for integrable models can be compared to the importance of appropriate choice of coordinates in mechanics. Quantum group Heisenberg subalgebra are well-known and

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used in the case of the homogeneous grading [3, 5, 12]. In this note, we propose a version of the Heisenberg subalgebra inside a quantum universal enveloping algebra in the principle [8, 5, 3, 12] grading case.

2. Principle Heisenberg Subalgebra in Quantum Group

Consider the Lie algebra sl_2 generated by the elements $\{h, x_\pm\}$ with standard relations [6]

$$[h, x_\pm] = \pm 2x_\pm,$$

$$[x_+, x_-] = h.$$

Following related ideas of [2], let us define the elements

$$\hat{x}_+ \equiv \Phi(h)\, x_+, \tag{1}$$

$$\hat{x}_- \equiv x_- \Phi(h), \tag{2}$$

$$\hat{h} \equiv h,$$

where $\Phi(h)$ is an invertible function of h. Then, we substitute (1)-(2) to the commutation relations of the quantum group [1] $U_q(sl_2)$ in the Drinfeld-Jimbo form [3, 5] to obtain

$$[\hat{h}, \hat{x}_\pm] = \pm 2\hat{x}_\pm,$$

$$[\hat{x}_+, \hat{x}_-] = [\hat{h}]_q \equiv \frac{q^{\hat{h}} - q^{-\hat{h}}}{q - q^{-1}}.$$

Then, assuming the formulas

$$\Phi(h)x_+ = x_+\Phi(h+2),$$

$$x_-\Phi(h) = \Phi(h+2)x_-,$$

and the form of the Casimir operator [1]

$$C = 2x_+x_- + \frac{1}{2}h^2 - h,$$

to express x_+x_- and x_-x_+ via C and h. One can solve the equation

$$\Phi(h)\, x_+x_-\Phi(h) - x_-\Phi^2(h)\, x_+ = [\,\hat{h}\,]_q\,,$$

to obtain $\Phi(h)$ from

$$\Phi^2(h)\left[C - \frac{h(h-2)}{2}\right] - \Phi^2(h+2)\left[C - \frac{h(h+2)}{2}\right] = 2\,\frac{q^h - q^{-h}}{q - q^{-1}}\,,$$

in terms of C and h.

Introducing the affinization of the Lie algebra sl_2 with $\lambda \in \mathbb{C}$ as in [6], we have for the generators of the zero-level Heisenberg subalgebra in the affine $\widehat{sl_2}$

$$E_+(\lambda) \equiv x_+ + \lambda x_-, \tag{3}$$

$$E_-(\lambda) \equiv x_- + \lambda^{-1}x_+ = \lambda^{-1}E_+(\lambda). \tag{4}$$

Then the operator

$$F(\lambda, \zeta) \equiv A(\lambda, \zeta)h + B(\lambda, \zeta)(x_+ - \lambda x_-),$$

with

$$A(\lambda, \zeta) \equiv -\sum_{j\in\mathbb{Z}} \zeta^{-2j}\lambda^j,$$

$$B(\lambda, \zeta) \equiv \sum_{j\in\mathbb{Z}} \zeta^{-2j-1}\lambda^j,$$

is an eigenoperator with respect to (3)-(4) for $\zeta \in \mathbb{C}$ [6].

Now, let us define for the quantum Heisenberg subalgebra $\mathcal{H}_q(\widehat{sl_2})$ generated by the elements $\{\widehat{E}_+, \widehat{E}_-\}$,

$$\widehat{E}_\pm(\lambda, h) = \Phi(h) E_\pm(\lambda) \Phi^{-1}(h), \tag{5}$$

so that

$$\left[\widehat{E}_+(\lambda, h), \widehat{E}_-(\lambda, h)\right] = 0,$$

and put

$$\widehat{F}(\lambda, \zeta, h) = \Phi(h) F(\lambda, \zeta) \Phi^{-1}(h),$$

thereof $\widehat{F}(\lambda, \zeta)$ are eigenoperators for $\widehat{E}_\pm$, $\zeta \in \mathbb{C}$. In terminology of ordinary Lie algebras, constructed elements $\{\widehat{E}_+, \widehat{E}_-\}$ constitute the principle Heisenberg subalgebra inside the quantized universal enveloping algebra $U_q(\widehat{sl_2})$. Similar considerations are possible for the case of homogeneous [6, 12] grading of $U_q(\widehat{sl_2})$.

3. Applications

Using the construction of previous section, we can define vertex operators [6, 7, 5]

$$V(z) \equiv \exp\left(\sum_{n=0}^{\infty} \widehat{E}_{+n} z^n\right) \exp\left(\sum_{n=0}^{\infty} \widehat{E}_{n} z^{-n}\right) z^n \partial_z^\alpha, \tag{6}$$

for $\alpha \in \mathbb{C}$, z being formal parameters. Suitable for the construction of a vertex operator representations [5] for corresponding quantum group. One could also relate this construction to quantum vertex algebras [9].

In [11], a group-theoretical [8] way to construct solutions to the affine Toda models was found. In particular, an algebraic origin of classical solitonic solutions was proposed. It is based on the existence of a Heisenberg subalgebra inside an affine Lie algebra underlying corresponding affine Toda model, and soliton vertex operators. In [12], we have studied the quantum group structure of the quantum soliton vertex operators for the sine-Gordon model. Those vertex operators corresponded to the homogeneous grading of $\hat{s}l_2$.

A general way to obtain the principle Heisenberg subalgebra proposed in these notes opens a way to study quantum vertex operators for the sine-Gordon model associated to the principle grading of $\hat{s}l_2$. Using the form of the quantum principle Heisenberg subalgebra, we can define vertex operators as in (6) and prove that they exhibit other properties of quantum vertex operators [12]. One can also generate solitonic specializations to the quantum Heisenberg operator solutions to the affine Toda models [10]. The group element [11] in the formal general solution in the solitonic specialization can be chosen in the from

$$g = \exp\left(z^{+}E_{+}\right)g_0 \exp\left(z^{-}E_{-}\right), \tag{7}$$

where $z^{\pm}$ are light-cone coordinates on the plane, and g_0 does not depend on $z^{\pm}$. In quantum case, we replace the generators $E_{\pm}$ (3)-(4) with the generators $\widehat{E}_{\pm}$ (5). The group element (similar to the classical one in [11]) is then given by

$$g_0 = \prod_{m=1}^{N} Q_m \exp\left(\widehat{F}(\lambda, \zeta_m, h)\right), \tag{8}$$

where $Q_m \in \mathbb{R}$, $m = 1, \ldots, N$ are some real constants, and $\zeta_m \in \mathbb{C}$ play a role of soliton rapidities. Due to the properties of the quantum principle Heisenberg subalgebra discussed in Section 2, it is easy to commute exponentials of the generators $\widehat{E}_{\pm}$ with the group element (8). Then the generators $\widehat{E}_{\pm}$ act on corresponding highest/lowers quantum group representation space vectors, and we obtain the quantum soliton-generating operators $\omega_m \widehat{F}(\lambda, \zeta_m, h)$, where $\omega_m \in \mathbb{C}$ are their eigenvalues with respect to $\widehat{E}_{\pm}$.

In [4], the higher grading generalizations for the conformal affine Toda models were considered. Both in principle and homogeneous grading of an underlying affine Lie algebra interesting models were obtained and solved reflecting in particular physical interactions between Toda and matter fields (associated to higher grading generators). The construction given here will allow to construct quantum versions to the solutions of the mentioned higher grading Toda systems.

In a forthcoming paper, we will extend the construction given in this notes to the cases of general Kac-Moody algebras.

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